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Group Fields, Gravity, and Angular Momentum

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The angular momentum properties of different homotopic sectors of a group field are examined. The results are applied to gravity, which is essentially a group field, and it is shown that the usual gravity kinks do not have angular momentum 1/2.

We shall consider *group fields*, that is to say, field theories for which the field or mapping φ maps R^3 into a Lie group G, with φ mapping the infinite boundary of R^3 into the group identity e. Space-time is assumed to have the topology of R^4, unless specified to the contrary. Mappings $R^3 \to S^3$, $R^3 \to SO(3)$, $R^3 \to SU(n)$ would be typical examples. We shall assume that $\pi_3(G) \approx Z$, so that there is a kink counting number. This is true in most cases of physical interest, including gravity, which can be regarded as a group field. The purpose of this paper is to clarify the way in which extrinsic angular momentum (e.a.m.) $\frac{1}{2}$ can arise for group fields. The main conclusion is that e.a.m. $\frac{1}{2}$ *cannot* arise for the usual kinks of the gravitational field.

We distinguish two sorts of angular momentum: intrinsic angular momentum, often called spin, and extrinsic angular momentum, often called orbital angular momentum. The intrinsic angular momentum for a field ψ is the tensor operator $\mathbb{S} = (S^i{}_j)$, $i, j = 1,2,3$, where $S^i{}_j$ is the infinitesimal

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generator for a rotation in the (i, j) plane and has the form

$$S^i_{\ j} = -i\hbar\psi^k S^i_{\ jk}{}^m \frac{\delta}{\delta\psi}{}^m$$

For gravity, the intrinsic angular momentum is an integer. This follows from the algebraic properties of the operator $\mathbb{S}$ for a tensor field of rank 2. Intrinsic angular momentum $\frac{1}{2}$ is common (the Dirac field being an example). It will be of no interest in this work. Instead, we shall be concerned with the extrinsic angular momentum, which is given by $\mathbb{L} = -i\hbar\mathbb{R}\times\nabla$. In certain kink field theories, field configurations belonging to the 1-kink sector may have e.a.m. $\frac{1}{2}$. In such cases, the e.a.m. $\frac{1}{2}$ arises if and only if the 2π-rotation loops in (the 1-kink sector of) field space are nontrivial, i.e., not deformable to a point.

The transformation properties of the fields under rotation are of prime importance in determining the existence/nonexistence of e.a.m. $\frac{1}{2}$. Let $R(t)\in SO(3)$, $0\leqslant t\leqslant 1$, denote a 2π-rotation loop. Since all 2π-rotation loops are homotopic to each other, we shall be specific and choose:

$$R(t) = \|R_{ij}(t)\| = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t & 0 \\ -\sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppressing the t, R, or R_{ij} will be used to denote rotation by an arbitrary angle.

The rotation of a vector $\mathbf{x}\in R^3$ can be represented by $x_i' = R_{ij}x_j$ or by $\mathbf{x}' = R\mathbf{x}$, for short. Symmetry considerations require that the transformed fields φ' be such that the field equations appear the same for an observer using the transformed coordinate system $\mathbf{x}'$ and the transformed field φ' as for an observer using the original $\mathbf{x}$ and φ. This can be ensured, for example, by using a scalar field which, by definition, transforms according to

$$\varphi'(\mathbf{x}') = \varphi(\mathbf{x})$$

This can be rewritten

$$\varphi'(\mathbf{x}) = \varphi(R^{-1}\mathbf{x})$$

Another alternative is to consider a vector field (such as the electric field) $\mathbf{E}(\mathbf{x})$ which transforms according to

$$E_i'(\mathbf{x}') = R_{ij}E_j(\mathbf{x})$$

The latter can be rewritten

$$E_i'(\mathbf{x}) = R_{ij}E_j(R^{-1}\mathbf{x})$$

To say that a field is *spherically symmetric* is to say that the field configuration is unchanged under rotation, i.e., $\mathbf{E}^\circ(\mathbf{x}) = \mathbf{E}(\mathbf{x})$, or $R_{ij}E_j(R^{-1}\mathbf{x}) = E_i(\mathbf{x})$. This is readily checked for the usual inverse square law field $\mathbf{E}(\mathbf{x}) = \mathbf{x}/r^3$:

$$R_{ij}E_j(R^{-1}\mathbf{x}) = R_{ij}R_{jk}^{-1}x_k/r^3 = x_i/r^3.$$

For a vector field that is spherically symmetric, a rotation loop in field space is a single point and therefore trivial.

Let us now consider a group field. To be specific, we choose the range to be a 3-sphere, S^3. The latter can be parametrized by four real variables $(\varnothing_1, \varnothing_2, \varnothing_3, \varnothing_4)$ subject to $\varnothing_\mu\varnothing_\mu = 1$. Since $\pi_3(S^3) \approx Z$, there are kinks. Let $\varphi(\mathbf{x}) = (\varnothing_1, \varnothing_2, \varnothing_3, \varnothing_4)$ with $\varnothing_\mu = f_\mu(\mathbf{x})$, $\mu = 1,2,3,4$ be an example of a 1-kink mapping. Is e.a.m. $\frac{1}{2}$ possible for such a theory? This depends upon the transformation properties of the f_μ. Suppose that $(f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$ is defined to be a *vector* field and $f_4(\mathbf{x})$ to be a scalar field:

$$f_i'(\mathbf{x}) = R_{ij}f_j(R^{-1}\mathbf{x})$$

$$f_4'(\mathbf{x}) = f_4(R^{-1}\mathbf{x})$$

An example of a spherically symmetric 1-kink field exists, namely, the stereographic projection:

$$\varnothing_i \equiv f_i(\mathbf{x}) = 2ax_i/(r^2 + a^2), \qquad i = 1,2,3$$

$$\varnothing_4 \equiv f_4(\mathbf{x}) = (r^2 - a^2)/(r^2 + a^2)$$

Note that $a > 0$ is any constant and $r = |\mathbf{x}|$. In this case, given the assumed transformation properties of the f_μ,

$$f_i'(\mathbf{x}) = f_i(\mathbf{x})$$

$$f_4'(\mathbf{x}) = f_4(R^{-1}\mathbf{x}) = f_4(\mathbf{x})$$

so that the rotation loop is trivial. Any other (possibly nonspherically symmetric) 1-kink field will be homotopic to the stereographic projection. Thus all rotation loops in the 1-kink sector are trivial. E.a.m. $\frac{1}{2}$ is not present in such a theory.

A theory with fields ranging over S^3 has also been considered by Skyrme. In his theory, the $\{f_\mu(\mathbf{x})\}$ are taken to be the components of a four-dimensional *iso*vector and so all transform as *scalars* under spatial rotation (Skyrme, 1961, p. 128). Again, the $\{f_\mu(\mathbf{x})\}$ defined by the stereographic projection provide an example of a 1-kink field configuration, but their transformation properties are now

$$f_\mu'(\mathbf{x}) = f_\mu(R^{-1}\mathbf{x}), \qquad \mu = 1,2,3,4$$

which could be written (including the rotation path parameter t explicitly):

$$\varnothing_i' = R_{ij}^{-1}(t)\varnothing_j, \qquad i = 1,2,3$$

$$\varnothing_4' = \varnothing_4$$

This loop in the 1-kink sector of field space is not a single point. Its triviality/nontriviality is not *a priori* obvious. Williams and Zvengrowski (1977) have shown such a path to be nontrivial, so that the kink of Skyrme's theory has e.a.m. $\frac{1}{2}$.

In the case of Skyrme's theory, it is straightforward to show that the n-kink sector will admit e.a.m. $\frac{1}{2}$ for n odd but not for n even (Shastri, Williams, and Zvengrowski, 1980, p. 19). This is puzzling at first sight, since for a group field the different sectors of mapping space are homeomorphic images of each other. Consider a field theory with group G. Choosing a particular 1-kink field $g_1(\mathbf{x})$ (which might be the stereographic projection, for the S^3 case), the homeomorphisms between different sectors can be represented by $g_1^n(\mathbf{x})$ for different choices of integer n. For example, if $g(\mathbf{x})$ is a given 1-kink field, a 0-kink field $h(\mathbf{x})$ can be constructed according to $h(\mathbf{x}) = g(\mathbf{x})g_1^{-1}(\mathbf{x})$. Similarly, a rotation loop in the 1-kink sector (assuming a scalar transformation law) $g(R^{-1}(t)\mathbf{x})$, $0 \leqslant t \leqslant 1$, gives rise to a loop in the 0-kink sector: $g(R^{-1}(t)\mathbf{x})\cdot g_1^{-1}(\mathbf{x})$. The two loops have the same (non)triviality, but the latter is *not* a rotation loop. Instead, a rotation loop in the 0-kink sector would be given by $g(R^{-1}(t)\mathbf{x})\cdot g_1^{-1}(R^{-1}(t)\mathbf{x})$. Since g and g_1 are homotopic ($g \sim g_1$), such a rotation loop is easily shown to be trivial:

$$g(R^{-1}(t)\mathbf{x})\cdot g_1^{-1}(R^{-1}(t)\mathbf{x}) \sim g_1(R^{-1}(t)\mathbf{x})\cdot g_1^{-1}(R^{-1}(t)\mathbf{x}) = e$$

where e is the group identity. Hence the 0-kink sector does not admit e.a.m. $\frac{1}{2}$.

Now consider the case of the gravitational field. This is essentially a group field. The kink is controlled by the orientation of the light cones and for the space-time $R^4 = R^3 \times R^1$ (with the R^1 contractible to a point) the homotopic classification of metrics is equivalent to classifying mappings

$R^3 \to P^3 = SO(3)$ with the infinite boundary of R^3 mapping onto the identity element of $SO(3)$. Consider the matrices $\|\eta_{\mu\nu}\| = \mathrm{diag}(1,1,1,-1)$ and $\|\delta_{\mu\nu}\| = \mathrm{diag}(1,1,1,1)$. The Minkowski metric has elements $\eta_{\mu\nu}$ and is an example of a 0-kink metric. An example of a 1-kink metric would be

$$g_{\mu\nu} = \delta_{\mu\nu} - 2\varnothing_\mu \varnothing_\nu$$

where the $\{\varnothing_\mu \equiv f_\mu(\mathbf{x})\}$ are given by the stereographic projection. This metric is spherically symmetric, rotating to produce a trivial loop so that e.a.m. $\frac{1}{2}$ is not possible. This result was noted previously by Finkelstein (1966, 1978). Let us examine this point in detail. As in the S^3 case, the transformation properties are crucial. Williams (1971) and Shastri, Williams, and Zvengrowski (1980) have claimed that this metric (and any other metric of odd kink number) has e.a.m. $\frac{1}{2}$. This claim is not valid. The former work assumes that the $\varnothing_\mu \equiv f_\mu(\mathbf{x})$ transform as scalars, thereby rotating the $\mathbf{x}$ but not the f_μ, and (equivalently) the latter work involves rotating the f_μ but not the $\mathbf{x}$. This would be valid for a theory akin to Skyrme's in which the fields [mapping into $SO(3)$, say] were scalars. However, the $g_{\mu\nu}$ are the components of a second-rank tensor field and transform under spatial rotations according to

$$g'_{ij}(\mathbf{x}) = R_{ik} R_{jm} g_{km}(R^{-1}\mathbf{x})$$

with i, j, k, m running over 1,2,3. A 2π-rotation loop for the 1-kink example is then

$$\begin{aligned}
&R_{ik}(t) R_{jm}(t)\left[\delta_{km} - 2f_k(R^{-1}(t)\mathbf{x}) f_m(R^{-1}(t)\mathbf{x})\right] \\
&\quad = R_{ik}(t) R_{jk} - 2R_{ik}(t) f_k(R^{-1}(t)\mathbf{x}) R_{jm}(t) f_m(R^{-1}(t)\mathbf{x}) \\
&\quad = \delta_{ij} - 2f_i(\mathbf{x}) f_j(\mathbf{x})
\end{aligned}$$

which is a single point in the space of 1-kink metrics. Hence e.a.m. $\frac{1}{2}$ is not possible.

The above argument was presented for a space-time $R^3 \times R^1$. The assumed boundary conditions allow R^3 to be compactified so that the argument is valid for a space-time $S^3 \times R^1$. Shastri, Williams, and Zvengrowski (1980) consider space-times of the form $M \times R^1$ where the (closed, connected, orientable) 3-manifold M is either type 1 [admitting degree 1 maps into $SO(3)$] or type 2 [admitting only even degree maps into $SO(3)$]. Type 2 manifolds include S^3, $S^1 \times S^2$, $S^1 \times S^1 \times S^1$ and spheres with handles. Metrics for type 2 manifolds can be written in the form

$g_{\mu\nu} = \delta_{\mu\nu} - 2\varnothing_\mu\varnothing_\nu$ and so the above analysis implies that for type 2 manifolds the gravitational field cannot have e.a.m. $\frac{1}{2}$.

This work does not depend on the actual Lagrangian of gravity, with its concomitant canonical constraints and dynamical equations, but operates at the more general level of generating functionals and covariant quantization. Interesting results concerning gravitational e.a.m. $\frac{1}{2}$ have been established by John Friedman and Rafael Sorkin (1980). Their analysis takes the actual Lagrangian into account though at the cost of constraining the gravitational field to admit Cauchy surfaces, and thus restricting it to a small part of the 0-kink sector.

The gravitational structures considered in the present work are quite different from those whose rotation gives rise to the e.a.m. $\frac{1}{2}$ of Friedman and Sorkin. The former arise in topologically trivial, metrically anomalous gravitational universes that have no Cauchy surfaces. The latter arise in metrically trivial, topologically anomalous gravitational universes. In the terminology of Finkelstein and Misner (1959), the former (metrical anomalies) are M geons and the latter (topological anomalies) are O geons.

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